Generalised Tribonacci hybrid quaternions

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Abstract

In this study, firstly we define a new sequence related to hybrid quaternions that has not been defined before and is called a generalised Tribonacci hybrid quaternion. Then, the basic definitions and basic features of this new sequence were examined. Finally, addition operation, the Binet formula, and some special types of generalised Tribonacci quaternions are presented.

Keywords: quaternions; hybrid quaternions; generalised Tribonacci sequence

1 Introduction

In the literature, many studies have been conducted on number sequences from the past to the present. Recently, the applications of these number sequences in different areas have accelerated. However, it has also been observed that there is an increase in the diversity of these number sequences. New sequences such as Leonardo, Narayana, Mersenne, Tribonacci, and Nickel numbers have been added to these number sequences, which started with Fibonacci [1, 10–13]. In this study, we will define generalized Tribonacci hybrid quaternions using the Tribonacci sequence.

The generalised Tribonacci sequence was first introduced in 1972 by Shannon and Horadam [5]. This sequence is the third-order recurrence relation, which is the most generalised form of numbers such as Tribonacci, Padovan-Perrin, Narayana, and third-order Jacobsthal. The generalised Tribonacci sequence is given as $\{V_n(V_0, V_1, V_2; r, s, t)\}$, where $r, s, t \in \mathbf{R}$ and $n \geq 0$. The n^{th} generalised Tribonacci number is defined by the relation $V_n = rV_{n-1} + sV_{n-2} + tV_{n-3}$ [1]. The most well-known of these sequences are as follows:

- Tribonacci numbers $\{T_n(0, 0, 1; 1, 1, 1)\}$,
- Padovan–Perrin numbers $\{P_n(0, 1, 0; 0, 1, 1)\}$,
- third-order Jacobsthal numbers $\{J_n(0, 1, 1; 1, 1, 2)\}$.

The characteristic equation of these numbers is $x^3 - rx^2 - sx - t = 0$ and the roots of the equation are

$$\alpha = \alpha(r, s, t) = \frac{r}{3} + A + B,$$

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$$w_1 = \frac{r}{3} + \epsilon A + \epsilon^2 B \quad w_2 = \frac{r}{3} + \epsilon^2 A + \epsilon B.$$

where $A = (\frac{r^3}{27} + \frac{rs}{6} + \frac{t}{2} + \sqrt{\Delta})^{1/3}$, $B = (\frac{r^3}{27} + \frac{rs}{6} + \frac{t}{2} - \sqrt{\Delta})^{1/3}$, $\Delta = \frac{r^3 t}{27} - \frac{r^2 s^2}{108} + \frac{rst}{6} - \frac{s^3}{27} + \frac{t^2}{4}$, and $\epsilon = \frac{-1}{2} + i\frac{\sqrt{3}}{2}$ [6].

The Binet formula for the generalised Tribonacci numbers is

$$V_n = \frac{P\alpha^n}{(\alpha - w_1)(\alpha - w_2)} - \frac{Qw_1^n}{(\alpha - w_1)(w_1 - w_2)} + \frac{Rw_2^n}{(\alpha - w_2)(w_1 - w_2)}$$

where [1]

$$\begin{aligned} P &= V_2 - (w_1 + w_2)V_1 + w_1 w_2 V_0, \\ Q &= V_2 - (\alpha + w_2)V_1 + \alpha w_2 V_0, \\ R &= V_2 - (\alpha w_1)V_1 + \alpha w_1 V_0. \end{aligned}$$

The nth generalised hybrid Tribonacci number HV_n is defined by the relation

$$HV_n = V_n + V_{n+1}\mathbf{i} + V_{n+2}\varepsilon + V_{n+3}\mathbf{h},$$

where V_n is the nth generalised Tribonacci number, and $\mathbf{i}, \varepsilon, \mathbf{h}$ are hybrid units [14]. Let HV_n be the generalised hybrid Tribonacci number. Then after some necessary calculations, one can obtain the following recurrence relation:

$$HV_n = rHV_{n-1} + sHV_{n-2} + tHV_{n-3}, \quad n \geq 3$$

with initial conditions:

$$\begin{aligned} HV_0 &= a + b\mathbf{i} + c\varepsilon + (rc + sb + ta)\mathbf{h}, \\ HV_1 &= b + c\mathbf{i} + (rc + sb + ta)\varepsilon + \left(\left(r^2 + s\right)c + (rs + t)b + rsa\right)\mathbf{h}, \\ HV_2 &= c + (rc + sb + ta)\mathbf{i} + \left(\left(r^2 + s\right)c + (rs + t)b + rsa\right)\varepsilon \\ &\quad + \left(\left(r^3 + 2rs + t\right)c + \left(r^2 s + s^2 + rs\right)b + \left(r^2 s + st\right)a\right)\mathbf{h}. \end{aligned}$$

The generalised Tribonacci quaternion is defined as

$$Q_{v,n} = V_n + V_{n+1}i + V_{n+2}j + V_{n+3}k, \quad n \geq 0,$$

where V_n is the n-th generalised Tribonacci number [1]. The subsequent iterative relation is derived after some straightforward calculations:

$$Q_{v,n} = rQ_{v,n-1} + sQ_{v,n-2} + tQ_{v,n-3}, \quad n \geq 3.$$

The Binet formula for generalised Tribonacci quaternions

$$Q_{v,n} = \frac{P\underline{\alpha}\alpha^n}{(\alpha - w_1)(\alpha - w_2)} - \frac{Q\underline{w_1}w_1^n}{(\alpha - w_1)(w_1 - w_2)} + \frac{R\underline{w_2}w_2^n}{(\alpha - w_2)(w_1 - w_2)}$$

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where $\underline{\alpha} = 1 + \alpha i + \alpha^2 j + \alpha^3 k$, $\underline{w_1} = 1 + w_1 i + w_1^2 j + w_1^3 k$, and $\underline{w_2} = 1 + w_2 i + w_2^2 j + w_2^3 k$ [1]. In addition to this, the generating function for these quaternions [1]

$$g(x) = \frac{Q_{v,0} + (Q_{v,1} - rQ_{v,0})x + (Q_{v,2} - rQ_{v,1} - sQ_{v,0})x^2}{1 - rx - sx^2 - tx^3}.$$

Some identities about Generalised Tribonacci quaternions are as follows [1]:

- $\|Q_{v,n}\|^2 = V_n^2 + V_{n+1}^2 + V_{n+2}^2 + V_{n+3}^2$,
- $Q_{v,3n} = \sum_{l=0}^{n} \sum_{m=0}^{l} \binom{n}{l}\binom{l}{m} r^m s^{l-m} t^{n-l} Q_{v,l+m}$.

Tribonacci numbers and Tribonacci quaternions with negative indices are defined as follows.

DEFINITION 1.1 ([7]).
Let $T'_n = T_{-n}$, then for $n \geq 2$,

$$T'_n = T'_{n-3} - T'_{n-2} - T'_{n-1}.$$

DEFINITION 1.2 ([2]).
If we denote by $Q_{T',n}$ Tribonacci quaternions with negative indices $Q_{T,-n}$, then for $n \geq 2$,

$$Q_{T',n} = T'_n + iT'_{n-1} + jT'_{n-2} + kT'_{n-3}.$$

Hybrid quaternions were introduced by Dağdeviren in [8] as a new number system. A hybrid quaternion is

$$HQ = Q_0 + \mathbf{i}Q_1 + \boldsymbol{\varepsilon}Q_2 + \mathbf{h}Q_3$$

such that Q_0, Q_1, Q_2 and Q_3 are quaternions. The other presentation of a hybrid quaternion is

$$HQ = H_0 + iH_1 + jH_2 + kH_3$$

such that H_0, H_1, H_2, and H_3 are hybrid numbers. Sum, subtraction, and multiplication operations can be easily done with the information up to here. In the literature there are many other works about hybrid quaternions, we can refer to [9].

In this study, in the second section we introduce a new sequence associated with hybrid quaternions, named the generalised Tribonacci hybrid quaternion, which is previously unstudied. Then, we explore the fundamental definitions and essential properties of this newly established sequence. Furthermore, we present the addition operation, derive the Binet formula, and discuss several specific variants of the Generalised Tribonacci quaternions. In the last section, we finally give some features about some special types of generalised Tribonacci quaternions.

2 Generalised Tribonacci hybrid quaternions

DEFINITION 2.1
A generalised Tribonacci Hybrid quaternion can be defined in two equivalent ways:

$$\begin{aligned} HQ_{v,n} &= Q_{v,n} + Q_{v,n+1}\mathbf{i} + Q_{v,n+2}\varepsilon + Q_{v,n+3}\mathbf{h} \\ &= H_{v,n} + H_{v,n+1}i + H_{v,n+2}j + H_{v,n+3}k \end{aligned}$$

where $Q_{v,n}$ is n^{th} generalised Tribonacci quaternion and $H_{v,n}$ is n^{th} generalised Tribonacci Hybrid number.

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Using the equation $Q_{v,n} = rQ_{v,n-1} + sQ_{v,n-2} + tQ_{v,n-3}$ we can obtain

$$\begin{aligned} HQ_{v,n} &= Q_{v,n} + Q_{v,n+1}\mathbf{i} + Q_{v,n+2}\varepsilon + Q_{v,n+3}\mathbf{h} \\ &= rQ_{v,n-1} + sQ_{v,n-2} + tQ_{v,n-3} + \mathbf{i}(rQ_{v,n} + sQ_{v,n-1} + tQ_{v,n-2}) \\ &+ \varepsilon(rQ_{v,n+1} + sQ_{v,n} + tQ_{v,n-1}) + \mathbf{h}(rQ_{v,n+2} + sQ_{v,n+1} + tQ_{v,n}) \\ &= rHQ_{v,n-1} + sHQ_{v,n-2} + tHQ_{n-3} \end{aligned}$$

where $n \geq 3$.

THEOREM 2.2
The Binet formula for n^{th} generalised Tribonacci Hybrid quaternion is

$$HQ_{v,n} = \frac{P\underline{\alpha}\alpha^*\alpha^n}{(\alpha - w_1)(\alpha - w_2)} - \frac{Q\underline{w_1}w_1^*w_1^n}{(\alpha - w_1)(w_1 - w_2)} + \frac{R\underline{w_2}w_2^*w_2^n}{(\alpha - w_2)(w_1 - w_2)}$$

where $\alpha^* = 1 + \alpha\mathbf{i} + \alpha^2\varepsilon + \alpha^3\mathbf{h}$, $w_1^* = 1 + w_1\mathbf{i} + w_1^2\varepsilon + w_1^3\mathbf{h}$ and $w_2^* = 1 + w_2\mathbf{i} + w_2^2\varepsilon + w_2^3\mathbf{h}$

PROOF. We use quaternion coefficient form to proof,

$$\begin{aligned} HQ_{v,n} &= Q_{v,n} + Q_{v,n+1}\mathbf{i} + Q_{v,n+2}\varepsilon + Q_{v,n+3}\mathbf{h} \\ &= \frac{P\underline{\alpha}\alpha^n}{(\alpha - w_1)(\alpha - w_2)} - \frac{Q\underline{w_1}w_1^n}{(\alpha - w_1)(w_1 - w_2)} + \frac{R\underline{w_2}w_2^n}{(\alpha - w_2)(w_1 - w_2)} \\ &+ \frac{P\underline{\alpha}\alpha^{n+1}}{(\alpha - w_1)(\alpha - w_2)}\mathbf{i} - \frac{Q\underline{w_1}w_1^{n+1}}{(\alpha - w_1)(w_1 - w_2)}\mathbf{i} + \frac{R\underline{w_2}w_2^{n+1}}{(\alpha - w_2)(w_1 - w_2)}\mathbf{i} \\ &+ \frac{P\underline{\alpha}\alpha^{n+2}}{(\alpha - w_1)(\alpha - w_2)}\varepsilon - \frac{Q\underline{w_1}w_1^{n+2}}{(\alpha - w_1)(w_1 - w_2)}\varepsilon + \frac{R\underline{w_2}w_2^{n+2}}{(\alpha - w_2)(w_1 - w_2)}\varepsilon \\ &+ \frac{P\underline{\alpha}\alpha^{n+3}}{(\alpha - w_1)(\alpha - w_2)}\mathbf{h} - \frac{Q\underline{w_1}w_1^{n+3}}{(\alpha - w_1)(w_1 - w_2)}\mathbf{h} + \frac{R\underline{w_2}w_2^{n+3}}{(\alpha - w_2)(w_1 - w_2)}\mathbf{h} \\ &= \frac{P\underline{\alpha}\alpha^*\alpha^n}{(\alpha - w_1)(\alpha - w_2)} - \frac{Q\underline{w_1}w_1^*w_1^n}{(\alpha - w_1)(w_1 - w_2)} + \frac{R\underline{w_2}w_2^*w_2^n}{(\alpha - w_2)(w_1 - w_2)}. \end{aligned}$$

□

THEOREM 2.3
The generating function for the generalised Tribonacci Hybrid quaternion $HQ_{v,n}$ is

$$g_{HQ_{v,n}}(x) = \frac{HQ_{v,0} + (HQ_{v,1} - rHQ_{v,0})x + (HQ_{v,2} - rHQ_{v,1} - sHQ_{v,0})x^2}{1 - rx - sx^2 - tx^3}.$$

PROOF. We can write the following equation from the definition of the generating function

$$g_{HQ_{v,n}}(x) = HQ_{v,0} + HQ_{v,1}x + HQ_{v,2}x^2 + HQ_{v,3}x^3 + \cdots. \tag{1}$$

From this we can also write the following

$$-rxg_{HQ_{v,n}}(x) = -rHQ_{v,0}x - rHQ_{v,1}x^2 - rHQ_{v,2}x^3 - rHQ_{v,3}x^4 - \cdots, \tag{2}$$

$$-sx^2g_{HQ_{v,n}}(x) = -sHQ_{v,0}x^2 - sHQ_{v,1}x^3 - sHQ_{v,2}x^4 - sHQ_{v,3}x^5 - \cdots, \tag{3}$$

$$-tx^3g_{HQ_{v,n}}(x) = -tHQ_{v,0}x^3 - tHQ_{v,1}x^4 - tHQ_{v,2}x^5 - tHQ_{v,3}x^6 - \cdots. \tag{4}$$

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Since $HQ_{v,3}x^3 - rHQ_{v,2}x^3 - sHQ_{v,1}x^3 - tHQ_{v,0}x^3 = 0$, $HQ_{v,4}x^4 - rHQ_{v,3}x^4 - sHQ_{v,2}x^4 - tHQ_{v,1}x^4 = 0$, $\cdots$, if we sum these equations 1, 2, 3, and 4 side by side, then

$$(1 - rx - sx^2 - tx^3)g_{HQ_{v,n}}(x) = HQ_{v,0} + (HQ_{v,1} - rHQ_{v,0})x + (HQ_{v,2} - rHQ_{v,1} - sHQ_{v,0})x^2.$$

As a result, the proof is complete. □

We define Tribonacci Hybrid quaternions with negative indices as follows.

DEFINITION 2.4
For $n \geq 2$, let $HQ_{T',n} = HQ_{T,-n}$. Then,

$$HQ_{T',n} = Q_{T',n} + \mathbf{i}Q_{T',n-1} + \varepsilon Q_{T',n-2} + \mathbf{h}Q_{T',n-3}.$$

We give a property concerning the sums of generalised Tribonacci Hybrid quaternions.

THEOREM 2.5
Let $HQ_{v,n}$ be n^{th} generalised Tribonacci Hybrid quaternion, then

$$\sum_{m=0}^{n} HQ_{v,m} = \frac{P\underline{\alpha}\alpha^*(1-\alpha^{n+1})}{(\alpha - w_1)(\alpha - w_2)(1-\alpha)} - \frac{Q\underline{w_1}w_1^*(1-w_1^{n+1}))}{(\alpha - w_1)(w_1 - w_2)(1-w_1)} + \frac{R\underline{w_2}w_2^*(1-w_2^{n+1})}{(\alpha - w_2)(w_1 - w_2)(1-w_2)}$$

where α, w_1 and w_2 are roots of $x^3 - rx^2 - sx - t = 0$.

PROOF. The proof can be easily done using the Binet formula and the property $1 + a + a^2 + \cdots + a^n = \frac{1-a^{n+1}}{1-a}$. □

3 Some properties about some special types of generalised Tribonacci quaternions

In this section we will give some properties for some special types of generalised Tribonacci Hybrid quaternions, such as Tribonacci, Padovan, etc.

THEOREM 3.1
Let $H_{T,n}$, $HQ_{T,n}$, $HQ^*_{T,n}$ and $HQ_{L,n}$ be n^{th} Tribonacci Hybrid number, n^{th} Tribonacci Hybrid quaternion, conjugate of n^{th} Tribonacci Hybrid quaternion and n^{th} Tribonacci–Lucas Hybrid quaternion, respectively. The following hold.

i) $HQ_{T,n} + HQ^*_{T,n} = 2H_{T,n}$,
ii) $HQ_{L,n} = HQ_{T,n} + 2HQ_{T,n-1} + 3HQ_{T,n-2}$.

PROOF.

i) Let $Q_{T,n}$ and T_n be n^{th} Tribonacci quaternion and Tribonacci numbers, respectively. If we use the eq $Q_{T,n} + Q^*_{T,n} = 2T_n$ [2], then

$$\begin{aligned} HQ_{T,n} + HQ^*_{T,n} &= Q_{T,n} + \mathbf{i}Q_{T,n+1} + \varepsilon Q_{T,n+2} + \mathbf{h}Q_{T,n+3} \\ &\quad + Q^*_{T,n} + \mathbf{i}Q^*_{T,n+1} + \varepsilon Q^*_{T,n+2} + \mathbf{h}Q^*_{T,n+3} \\ &= 2T_n + \mathbf{i}2T_{n+1} + \varepsilon 2T_{n+2} + \mathbf{h}2T_{n+3} \\ &= 2H_{T,n}. \end{aligned}$$

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ii) The proof can be performed easily from the property $Q_{L,n} = Q_{T,n} + 2Q_{T,n-1} + 3Q_{T,n-2}$ [2] where $Q_{L,n}$ and $Q_{T,n}$ are n^{th} Tribonacci quaternion and n^{th} Tribonacci–Lucas quaternion, respectively. □

We give some properties of Tribonacci hybrid quaternions related to finite sums of sequences of interest.

THEOREM 3.2
Let $HQ_{T,n}$ be n^{th} Tribonacci Hybrid quaternion and $\{V_n(1,0,1;1,1,1)\}$, then

i)

$$\sum_{m=0}^{n} HQ_{T,m} = \frac{HQ_{T,n+2} + HQ_{T,n} - HQ'_{T,1} - HQ_{T,1}}{2},$$

ii)

$$\sum_{m=0}^{n} HQ_{T,2m} = \frac{HQ_{T,2n+1} + HQ_{T,2n} - HQ_{V,0}}{2},$$

iii)

$$\sum_{m=0}^{n} HQ_{T,2m+1} = \frac{HQ_{T,2n+2} + HQ_{T,2n+1} - HQ_{V,1}}{2}.$$

PROOF.

i) Firstly we express this summation clearly,

$$\sum_{m=0}^{n} HQ_{T,m} = \sum_{m=0}^{n} Q_{T,m} + \mathbf{i}\sum_{m=0}^{n} Q_{T,m+1} + \varepsilon \sum_{m=0}^{n} Q_{T,m+2} + \mathbf{h}\sum_{m=0}^{n} Q_{T,m+3}. \tag{5}$$

Now, if we use the equation $\sum_{m=0}^{n} Q_{T,m} = \frac{Q_{T,n+2}+Q_{T,n}+Q_{T,0}-Q_{T,2}}{2}$, we obtain the following equations:

$$\sum_{m=0}^{n} Q_{T,m+1} = \frac{Q_{T,n+2} + Q_{T,n} + Q_{T,0} - Q_{T,2}}{2} + Q_{T,n+1} - Q_{T,0},$$

$$\sum_{m=0}^{n} Q_{T,m+2} = \frac{Q_{T,n+2} + Q_{T,n} + Q_{T,0} - Q_{T,2}}{2} + Q_{T,n+2} + Q_{T,n+1} - Q_{T,0} - Q_{T,1},$$

$$\sum_{m=0}^{n} Q_{T,m+3} = \frac{Q_{T,n+2} + Q_{T,n} + Q_{T,0} - Q_{T,2}}{2} + Q_{T,n+3} + Q_{T,n+2} + Q_{T,n+1} - Q_{T,0} - Q_{T,1} - Q_{T,2}.$$

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By performing the appropriate operations, we obtain the following equations:

$$\sum_{m=0}^{n} Q_{T,m} = \frac{Q_{T,n+2} + Q_{T,n} - Q'_{T,1} - Q_{T,1}}{2},$$

$$\sum_{m=0}^{n} Q_{T,m+1} = \frac{Q_{T,n+3} + Q_{T,n+1} - Q_{T,0} - Q_{T,2}}{2},$$

$$\sum_{m=0}^{n} Q_{T,m+2} = \frac{Q_{T,n+4} + Q_{T,n+2} - Q_{T,1} - Q_{T,3}}{2},$$

$$\sum_{m=0}^{n} Q_{T,m+3} = \frac{Q_{T,n+5} + Q_{T,n+3} - Q_{T,2} - Q_{T,4}}{2}.$$

If we substitute these expressions in equation 5, the proof is completed.

ii) Similar to the proof in the first item, when we write this sum explicitly, we observe the sums $\sum_{m=0}^{n} Q_{T,2m}$, $\sum_{m=0}^{n} Q_{T,2m+1}$, $\sum_{m=0}^{n} Q_{T,2m+2}$, and $\sum_{m=0}^{n} Q_{T,2m+3}$. If we use the equations $\sum_{m=0}^{n} Q_{T,2m} = \frac{Q_{T,2n+1}+Q_{T,2n}-Q_{V,0}}{2}$ and $\sum_{m=0}^{n} Q_{T,2m+1} = \frac{Q_{T,2n+2}+Q_{T,2n+1}-Q_{V,1}}{2}$ to calculate these sums, we derive the following results:

$$\sum_{m=0}^{n} Q_{T,2m+2} = \frac{Q_{T,2n+1} + Q_{T,2n} - Q_{V,0}}{2} + Q_{T,2n+2} - Q_{T,0}$$

$$= \frac{Q_{T,2n+3} + Q_{T,2n+2} - Q_{V,2}}{2},$$

$$\sum_{m=0}^{n} Q_{T,2m+3} = \frac{Q_{T,2n+2} + Q_{T,2n+1} - Q_{V,1}}{2} + Q_{T,2n+3} - Q_{T,1}$$

$$= \frac{Q_{T,2n+4} + Q_{T,2n+3} - Q_{V,3}}{2}.$$

The proof is completed when we insert these statements in their places.

iii) It can be easily proved similar to the previous proof. □

THEOREM 3.3

Let $HQ_{P,n}$, $HQ^{*}_{P,n}$, and $H_{P,n}$ be n^{th} Padovan Hybrid quaternion, conjugate of n^{th} Padovan Hybrid quaternion, and n^{th} Padovan Hybrid number, respectively.

i) $HQ_{P,n}HQ^{*}_{P,n} = H^2_{P,n} + H^2_{P,n+1} + H^2_{P,n+2} + H^2_{P,n+3}$,
ii) $HQ_{P,n} - iHQ_{P,n+1} - jHQ_{P,n+2} - kHQ_{P,n+3} = H_{P,n} + H_{P,n+2} + H_{P,n+4} + H_{P,n+6}$,
iii) $\sum_{m=0}^{n} HQ_{P,3m} = HQ_{P,3n+2} - HQ_{P,-1}$.

PROOF. The first and second proofs can be easily reached from the properties of quaternion units.

iii) If we express Padovan Hybrid quaternions with Padovan quaternion coefficients and use the properties $\sum_{m=0}^{n} Q_{P,3m} = Q_{P,3n+2} - Q_{P,-1}$, $\sum_{m=0}^{n} Q_{P,3m+1} = Q_{P,3n+3} - Q_{P,0}$,

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$\sum_{m=0}^{n} Q_{P,3m+2} = Q_{P,3n+4} - Q_{P,1}$, and $\sum_{m=0}^{n} Q_{P,3m+3} = Q_{P,3n+5} - Q_{P,2}$ [3], then

$$\sum_{m=0}^{n} HQ_{P,3m} = Q_{P,3n+2} - Q_{P,-1} + \mathbf{i}(Q_{P,3n+3} - Q_{P,0}) + \varepsilon(Q_{P,3n+4} - Q_{P,1}) + \mathbf{h}(Q_{P,3n+5} - Q_{P,2})$$
$$= HQ_{P,3n+2} - HQ_{P,-1}.$$

□

THEOREM 3.4
Let $HQ_{J,n}$ and $HQ_{j,n}$ be n^{th} third-order Jacobsthal Hybrid quaternion and third-order Jacobsthal–Lucas Hybrid quaternion.

i) Let $\{V_n(2,-1,-1;2,1,1)\}$. For $n \geq 2$,

$$\frac{HQ_{j,n} - 4HQ_{j,n-2}}{3} = \begin{cases} HQ_{V,0}, & n \equiv 0 \ (\text{mod}3) \\ HQ_{V,1}, & n \equiv 1 \ (\text{mod}3) \\ HQ_{V,2}, & n \equiv 2 \ (\text{mod}3) \end{cases},$$

ii) For $n \geq 0$, $3HQ_{J,n} + HQ_{j,n} = 2^{n+1}(1 + 2i + 4j + 8k)(1 + 2\mathbf{i} + 4\varepsilon + 8\mathbf{h})$,

iii) For $n \geq 0$, $HQ_{j,n+1} + HQ_{j,n} = 3HQ_{J,n+2}$,

iv) Let $\{V_n^{(1)}(1,-4-5;2,1,1)\}$, $\{V_n^{(2)}(-2,-4,-2;2,1,1)\}$, and $\{V_n^{(3)}(-2,-1,-5;2,1,1)\}$. For $n \geq 0$,

$$\sum_{m=0}^{n} HQ_{j,m} = \begin{cases} HQ_{j,n+1} + HQ_{V^{(1)},0}, & n \equiv 0 \ (\text{mod}3) \\ HQ_{j,n+1} + HQ_{V^{(2)},0}, & n \equiv 1 \ (\text{mod}3) \\ HQ_{j,n+1} + HQ_{V^{(3)},0}, & n \equiv 2 \ (\text{mod}3) \end{cases}$$

PROOF.

i) If we open this expression according to the quaternion coefficients and use the following property [4], the result is easily seen.

$$\frac{Q_{j,n} - 4Q_{j,n-2}}{3} = \begin{cases} 2 - i - j + 2k, & n \equiv 0 \ (\text{mod}3) \\ -1 - i + 2j - k, & n \equiv 1 \ (\text{mod}3) \\ -1 + 2i - j - k, & n \equiv 2 \ (\text{mod}3) \end{cases}$$

ii) Let $n \geq 0$.

$$3HQ_{J,n} + HQ_{j,n} = (3Q_{J,n} + Q_{j,n}) + \mathbf{i}(3Q_{J,n+1} + Q_{j,n+1})$$
$$= +\varepsilon(3Q_{J,n+2} + Q_{j,n+2}) + \mathbf{h}(3Q_{J,n+3} + Q_{j,n+3}).$$

By using the property $3Q_{J,n} + Q_{j,n} = 2^{n+1}(1 + 2i + 4j + 8k)$ [4], we obtain the following:

$$3HQ_{J,n} + HQ_{j,n} = 2^{n+1}(1 + 2i + 4j + 8k) + \mathbf{i}2^{n+2}(1 + 2i + 4j + 8k)$$
$$+ \varepsilon 2^{n+3}1 + 2i + 4j + 8k) + \mathbf{h}2^{n+4}(1 + 2i + 4j + 8k)$$
$$= 2^{n+1}(1 + 2i + 4j + 8k)(1 + 2\mathbf{i} + 4\varepsilon + 8\mathbf{h}).$$

iii) It is easy to see the result if we use the property $Q_{j,n+1} + Q_{j,n} = Q_{J,n+2}$ [4] and express with quaternion coefficients.

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iv) We will use the following property [4] to prove this:

$$\sum_{m=0}^{n} jQ_m = \begin{cases} Q_{j,n+1} + (1 - 4i - 5j - 7k), & n \equiv 0 \ (\text{mod}3) \\ Q_{j,n+1} - 2(1 + 2i + j + 5k), & n \equiv 1 \ (\text{mod}3) \\ Q_{j,n+1} - (2 + i + 5j + 10k), & n \equiv 2 \ (\text{mod}3) \end{cases}$$

If we expand this statement in terms of quaternion coefficients, we need to calculate the sums $\sum_{m=0}^{n} Q_{j,m+1}$, $\sum_{m=0}^{n} Q_{j,m+2}$, and $\sum_{m=0}^{n} Q_{j,m+3}$. For $n \equiv 0$ (mod3),

$$\begin{aligned} \sum_{m=0}^{n} Q_{j,m+1} &= \sum_{m=0}^{n+1} Q_{j,m} - Q_{j,0} \\ &= Q_{j,n+2} - (2 + 4i + 2j + 10k) - (2 + i + 5j + 10k) \\ &= Q_{j,n+2} - (4 + 5i + 7j + 20k), \end{aligned}$$

$$\begin{aligned} \sum_{m=0}^{n} Q_{j,m+2} &= \sum_{m=0}^{n+2} Q_{j,m} - Q_{j,0} - Q_{j,1} \\ &= Q_{j,n+3} - (2 + i + 5j + 10k) - (2 + i + 5j + 10k) - (1 + 5i + 10j + 17k) \\ &= Q_{j,n+3} - (5 + 7i + 20j + 37k), \end{aligned}$$

$$\begin{aligned} \sum_{m=0}^{n} Q_{j,m+3} &= \sum_{m=0}^{n+3} Q_{j,m} - Q_{j,0} - Q_{j,1} - Q_{j,2} \\ &= Q_{j,n+4} + (1 + 4i - 5j - 7k) - (2 + i + 5j + 10k) - (1 + 5i + 10j + 17k) \\ &\quad - (5 + 10i + 17j + 37k) \\ &= Q_{j,n+4} - (7 + 20i + 37j + 71k). \end{aligned}$$

Let $\{V_n^{(1)}(1, -4, -5; 2, 1, 1)\}$, then

$$\sum_{m=0}^{n} HQ_{j,m} = HQ_{j,n+1} + HQ_{V^{(1)},0}.$$

Other cases can be proven in a similar way. □

4 Conclusion

In this study, we investigated the most general form of Tribonacci numbers within the framework of the hybrid quaternion number system. We examined various properties and identities of certain special Tribonacci-type numbers in the context of hybrid quaternions. This study is expected to serve as a foundation for future research on third-order number sequences.

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