PARAMETER ESTIMATION IN A BLACK-SCHOLES MODEL

by

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In this paper we discuss parameter estimation in black scholes model. A non-parametric estimation method and well known maximum likelihood estimator are considered. Our aim is to estimate the unknown parameters for stochastic differential equation with discrete time observation data. In simulation study we compare the non-parametric method with maximum likelihood method using stochastic numerical scheme named with Euler Maruyama.

Key words: *non-parametric estimation method, stochastic differential equations, black scholes model, maximum likelihood estimation method*

Introduction

When the stochastic differential equation is given with certain coefficients, it is easy to solve this equation with numerical methods, Euler Maruyama method, Milstein method, Runge Kutta method, *etc.* [1, 2]. However if we have only observed discrete data over any time interval it has difficulty. To reach diffusion and drift coefficients are required in this case, so we need estimation methods. [1, 3-5], mention about estimation methods for stochastic differential equations. A number of researchers have used non-parametric techniques [5, 6]. Non-parametric method is simple for to implement and estimate the coefficients, but it is reasonably if we have frequent data. Maximum likelihood method [10] is other estimation method which is more efficient than non-parametric method.

In many practical cases where a diffusion process has been observed at discrete time points an explicit likelihood function is rarely available. These types of data have recently received great interest. A simple method to obtain an estimator for discrete time is constructing from the stale data, an approximation to the estimator found in the theory for continuous observations which is includes discrete time approximations to stochastic integrals. [8, 9], *etc.* study such estimators analytically, in general it is difficult but it can be done easily by numerical simulation.

In this paper our aim is discuss the differences between non-parametric estimation method and maximum likelihood estimation (MLE) method with numerical application in finance.

Non-parametric estimation method

We consider $X(t)$ diffusion process, which satisfies the stochastic differential equation:

$$\mathrm{d}X(t) = \mu[X(t)]\,\mathrm{d}t + \sigma[X(t)]\,\mathrm{d}W(t) \tag{1}$$

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Under proper restrictions on μ, σ and arbitrary function ϕ, from [10] conditional expectation $E_t[\phi(X_{t+\Delta t},t)]$ in the Taylor series form can be written:

$$E_t[\phi(X_{t+\Delta t},t)] = \phi(X_t,t) + \mathcal{L}\phi(X_t,t)\Delta t + \frac{1}{2}\mathcal{L}^2\phi(X_t,t)(\Delta t)^2 + \ldots + \frac{1}{n!}\mathcal{L}^n\phi(X_t,t)(\Delta t)^n + O\left[(\Delta t)^{n+1}\right] \tag{2}$$

where

$$\mathcal{L}\phi(x,t) = \lim_{\zeta \downarrow t}\frac{E\left[\phi(X_\zeta,\zeta)\mid X_t = x\right] - \phi(x,t)}{\zeta - t} = \frac{\partial\phi(x,t)}{\partial t} + \frac{\partial\phi(x,t)}{\partial x}\mu(x) + \frac{1}{2}\frac{\partial^2\phi(x,t)}{\partial x^2}\sigma^2(x) \tag{3}$$

from eq. (2):

$$\mathcal{L}\phi(X_t,t) = \frac{1}{\Delta t}E_t[\phi(X_{t+\Delta t},t+\Delta t) - \phi(X_t,t)] - \frac{1}{2}\mathcal{L}^2\phi(X_t,t)\Delta t - \ldots \tag{4}$$

is obtained. Taking the first term on the right side of eq. (4) and ignoring other terms as an error we obtain first order approximate for $\mathcal{L}\phi$. So we can write:

$$\mathcal{L}\phi(X_t,t) = \frac{1}{\Delta t}E_t[\phi(X_{t+\Delta t},t+\Delta t) - \phi(X_t,t)] + O(\Delta t) \tag{5}$$

If we want approximate a specific $\rho(x,t)$ function, we need only to specify ϕ function which provide $\mathcal{L}\phi(X_t,t) = \rho(x,t)$. To find $\mu(X_t)$ coefficient we take $\phi_1(x,t) \equiv x$ and to find $\sigma(X_t)$ coefficient, $\phi_2(x,t) \equiv (x - X_t)^2$ is determined in [5]. Then from eq. (3):

$$\mathcal{L}\phi_1(X_t,t) = \mu(X_t) \tag{6}$$

$$\mathcal{L}\phi_2(X_t,t) = \sigma^2(X_t) \tag{7}$$

are obtained. Accordingly:

$$\mu(X_t) = \frac{1}{\Delta t}E_t[X_{t+\Delta t} - X_t)] + O(\Delta t) \tag{8}$$

$$\sigma^2(X_t) = \frac{1}{\Delta t}E_t[(X_{t+\Delta t} - X_t)^2] + O(\Delta t) \tag{9}$$

are found.

If we want to find θ parameter vector via non-parametric estimation method in

$$\mathrm{d}X(t) = \mu\left[X(t);\theta\right]\mathrm{d}t + \sigma\left[X(t);\theta\right]\mathrm{d}W(t) \tag{10}$$

and we have $x_0, x_1, \ldots, x_N$ observed data of X_t at the respectively uniformly distributed times $t_i = i\Delta t$ for $i = 0,1,\ldots,N$ where $\Delta t = T/N$, from eqs. (8) and (9) we can find θ using following equations:

$$\sum_{i=0}^{N-1}\mu(t_i,x_i,\theta) = \frac{1}{\Delta t}\sum_{i=0}^{N-1}(x_{i+1} - x_i) \tag{11}$$

$$\sum_{i=0}^{N-1}\sigma^2(t_i,x_i,\theta)=\frac{1}{\Delta t}\sum_{i=0}^{N-1}(x_{i+1}-x_i)^2 \tag{12}$$

The MLE method

We consider eq. (10). Assume that for $j=0,1,2,...,N$, x_j are known, density of the initial case is $g_0(x_0|\theta)$ and transition probability density of (t_j,x_j) beginning from (t_{j-1},x_{j-1}) is $g(t_j,x_j|t_{j-1},x_{j-1};\theta)$. Then from [11], MLE of θ is value of θ which maximizes following joint density equation:

$$\mathcal{D}(\theta)=g_0(x_0\,|\,\theta)\prod_{j=1}^{N}g(t_j,x_j\,|\,t_{j-1},x_{j-1};\theta) \tag{13}$$

with $L(\theta)=-\ln[\mathcal{D}(\theta)]$ transformation we can rewrite eq. (13):

$$\mathrm{L}(\theta)=-\ln[g_0(x_0\,|\,\theta)]-\sum_{j=1}^{N}\ln[g(t_j,x_j\,|\,t_{j-1},x_{j-1};\theta)] \tag{14}$$

Our aim is find the minimum value of θ which make $L(\theta)$ function minimum. This proper θ value demonstrate with $\overline{\theta}$. From Euler Maruyama approximation scheme we can write:

$$x_{t_j}\approx x_{t_{j-1}}+\mu(t_{j-1},x_{t_{j-1}};\theta)\Delta t+g(t_{j-1},x_{t_{j-1}};\theta)\sqrt{\Delta t}\xi_j \tag{15}$$

where $\xi_i\sim N(0,1)$ and transition probability density is:

$$g(t_j,x_j\,|\,t_{j-1},x_{j-1};\theta)\approx\frac{1}{\sqrt{2\pi\delta_j^2}}\exp\left[\frac{-(x_j-\gamma_j)^2}{2\delta_j^2}\right] \tag{16}$$

where

$$\gamma_j=x_{t_{j-1}}+\mu(t_{j-1},x_{t_{j-1}};\theta)\Delta t$$

$$\delta_j=g(t_{j-1},x_{t_{j-1}};\theta)\sqrt{\Delta t}$$

We can approximate density function as in [11], for arbitrary value of θ, we can find solution of eq. (10) numerically. This process is repeated *M* times and we show this estimated values with $\overline{x_i}$. Then the transition density $g(t_j,x_j|t_{j-1},x_{j-1};\theta)$ can be estimated by the formula:

$$g^{(M)}(t_j,x_j\,|\,t_{j-1},x_{j-1};\theta)=\frac{1}{Mh}\sum_{i=1}^{M}K\left[\frac{x_j-\overline{x_i}}{h}\right] \tag{17}$$

where *K* is a non-negative kernel function which demonstrate following:

$$K(\vartheta)=\frac{1}{\sqrt{2\pi}}\exp\left[\frac{-\vartheta^2}{2}\right]$$

and h is bandwidth equal to

$$h=0.9\varepsilon M^{-1/5}$$

where

$$\varepsilon^2 = \frac{1}{M-1}\left[\sum_{i=1}^{M}\bar{x}_i^2 - \frac{1}{M}\left(\sum_{i=1}^{M}\bar{x}_i\right)^2\right]$$

Analysis of experimental data

In this section we tackle the YHOO stock. The stock data set is graphed for every month in fig. 1 between the years 2005-2015.

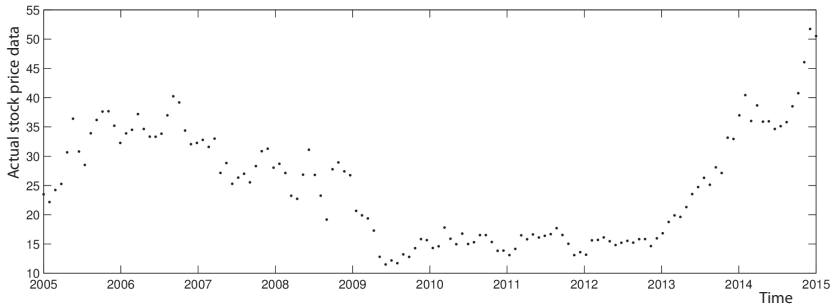


Figure 1. Monthly YHOO stock, between January 2005 to January 2015

We suit this data to stochastic differential equation named Black-Scholes model [12]:

$$\begin{aligned} X(t) &= \theta_1 X(t)\mathrm{d}t + \theta_2 X(t)\mathrm{d}W(t) \\ X(0) &= 23.49 \end{aligned} \tag{18}$$

where $X(t)$ is the stock price at time t and $\theta = [\theta_1, \theta_2]^T$ is determined. We need to estimate the parameter θ_1 and θ_2 using MLE and non-parametric parameter estimation methods. Firstly we estimate the θ_1 and θ_2 with MLE method. If we use MLE procedure we obtain the optimal values $\overline{\theta_1} = 0.0107$ and $\overline{\theta_2} = 0.1000$, approximately. Therefore, a reasonable stochastic differential equation based on the data for the YHOO stock for 132 months is:

$$\begin{aligned} \mathrm{d}X(t) &= 0.0107X(t)\mathrm{d}t + 0.1000X(t)\mathrm{d}W(t) \\ X(0) &= 23.49 \end{aligned} \tag{19}$$

Now we estimate the θ_1 and θ_2 with non-parametric estimation method. If we use non-parametric estimation method procedure for $N = 131$ and $\Delta t = 1$, we obtain the optimal values $\widehat{\theta_1} = 0.0084$ and $\widehat{\theta_2} = 0.0952$, approximately. Therefore, a reasonable stochastic differential equation based on the data for the YHOO stock for 132 months is:

$$\begin{aligned} \mathrm{d}X(t) &= 0.0084X(t)\mathrm{d}t + 0.0952X(t)\mathrm{d}W(t) \\ X(0) &= 23.49 \end{aligned} \tag{20}$$

The mean of stock price of actual data is $\mu = 24.8445$ and $(23.2291, 26.4600)$ gives its 95% confidence interval. Solving eqs. (19) and (20) with Euler Maruyama approximation method we obtain the mean of stock price forecasting with MLE is $\overline{\mu} = 24.4767$ and $(22.7030, 26.2504)$ gives its 95% confidence interval and the mean of stock price forecasting with non-parametric parameter estimation method is $\widehat{\mu} = 22.1242$ and $(20.5504, 23.6980)$ gives its 95% confidence interval, respectively.

In fig. 2, the simulated data, evaluated with MLE using the stochastic differential equation (SDE) model, and the monthly actual data are plotted for every month between the

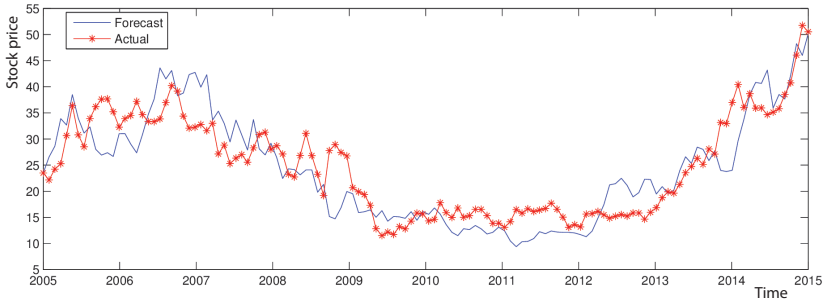


Figure 2. Estimated data using MLE and actual data of the YHOO stock, January 2005 to January 2015 (*for color image see journal web site*)

year 2005-2015. *Actual* holds real data for the YHOO stock, which is plotted as red straight lines. *Forecast* keeps MLE estimation using Euler Maruyama approximations, which is plotted as blue straight lines.

In fig. 3, the simulated data, evaluated with non-parametric parameter estimation using the SDE model, and the monthly actual data are plotted for every month between the year 2005-2015. *Actual* holds real data for the YHOO stock, which is plotted as red straight lines. *Forecast* keeps non-parametric estimation using Euler Maruyama approximations, which is plotted as blue straight lines.

All of the graphs indicate that the SDE model supplies sensible fit to the data.

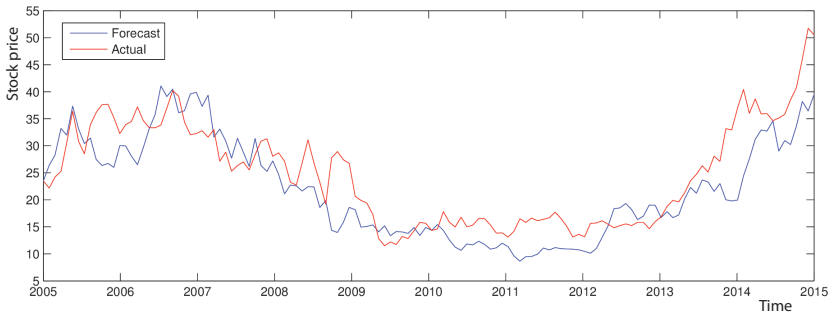


Figure 3. Estimated data using non-parametric estimation and actual data of the YHOO stock January 2005 to January 2015 (*for color image see journal web site*)

Conclusion

In this paper we have been concerned with the estimation of the θ_1 and θ_2 in the drift coefficient and in the diffusion coefficient, respectively, of a Black-Scholes model, when the observation data known. Using the YHOO stock data monthly between January 1, 2005 and January 1, 2015 MLE parameters $\overline{\theta_1}$, $\overline{\theta_2}$ and non-parametric estimation parameters $\widehat{\mu_1}$, $\widehat{\mu_2}$ are obtained. This obtained parameters written in the Black-Scholes model. Then applying Euler Maruyama method this stochastic differential equation with its initial value, the simulated solution is obtained for each estimation method at each time. After that, we compare actual data with numerical solutions for each estimation method. According to our results we can say that MLE method is have a good approximation to observation data via non-parametric estimation method.

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