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A Note on the Numerical Solution of Coupled Klein-Gordon Equations

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Abstract. This study presents the numerical solution of coupled Klein-Gordon equations, which models exchange of energy between two different components of one-dimensional nonlinear wave process. A composite numerical method based on the first order of accuracy finite different scheme and fixed point iteration is implemented to solve coupled Klein-Gordon equations with appropriate initial and boundary conditions. A test problem is employed and results of numerical experiments are presented with error analysis.
Keywords: Existence, Uniqueness, Weak solutions, Finite Difference, Fixed Point Theory.
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INTRODUCTION

The nonlinear single and coupled Klein-Gordon equations, play a fundamental role in various applications in physics and nonlinear theory. Many physical phenomena are described by these nonlinear equations. Khusnutdinova et. al. [1] considered coupled Klein-Gordon equations and discussed the exchange of energy between components of the system, Yomosa [2] generated the open states in deoxyribonucleic acid (DNA) double helices, Scott [3] studied the magnetic flux on Josephson junction transmission line, Seeger et al. [11] considered the movement of a slide dislocation in crystalline structure, and many other applications for nonlinear waves can be seen in the literature [4]-[8] and [13]-[17].

Various methods for finding soliton solutions of system of coupled Klein-Gordon equations are proposed by many scientists. Bilinear form and one-soliton solutions of this system is investigated in [10] and [12]. The sine-cosine method is used to obtain exact periodic and soliton solutions of coupled nonlinear systems in [18]. Global existence of solutions for the system is presented in [19].

The study of single and coupled Klein-Gordon equations has been a popular source of interest from a numerical perspective. For instance, both implicit and explicit finite-difference schemes were proposed for Klein-Gordon equations in [21]-[23]. In [22] a numerical technique is presented for radially symmetric solutions for conservative Klein-Gordon equations. Numerical solution of the damped Klein-Gordon equations is considered in [20], using variational method and finite element approximation.

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In this study, the initial-boundary value problem (IBVP)

$$\begin{cases} u_{tt} - u_{xx} = f_u(u,v), 0 < t < 1, 0 < x < \pi, \\ v_{tt} - c^2 v_{xx} = f_v(u,v), 0 < t < 1, 0 < x < \pi, \\ u(0,x) = \varphi_1(x), u_t(0,x) = \psi_1(x), 0 \le x \le \pi, \\ v(0,x) = \varphi_2(x), v_t(0,x) = \psi_2(x), 0 \le x \le \pi, \\ u(t,0) = u(t,l) = 0, 0 \le t \le 1, \\ v(t,0) = v(t,l) = 0, 0 \le t \le 1 \end{cases} \tag{1}$$

governed by coupled Klein-Gordon equations is considered. Here, $f(u,v)$ is a potential function for nonlinear coupling, c is the ratio for the acoustic velocities of the functions u, v, and $\varphi_i(x), \psi_i(x), i = 1,2$ are smooth functions. A unified numerical method based on the first order of accuracy finite difference scheme

$$\begin{cases} \tau^{-2}(u_{k+1} - 2u_k + u_{k-1}) + Au_{k+1} = f_k, \\ f_k = f(t_k, u_{k,k}), t_k = k\tau,\ 1 \le k \le N-1, N\tau = 1, \\ \tau^{-2}(v_{k+1} - 2v_k + v_{k-1}) + Av_{k+1} = g_k, \\ g_k = g(t_k, u_k, v_k), t_k = k\tau,\ 1 \le k \le N-1, N\tau = 1, \\ u_0 = \varphi_1, u_0' = (I + \tau^2 A)\frac{u_1 - u_0}{\tau} = \psi_1, \\ v_0 = \varphi_2, v_0' = (I + \tau^2 A)\frac{v_1 - v_0}{\tau} = \psi_2, \end{cases} \tag{2}$$

with a modification for nonlinear coupled system, and fixed point iteration is used for problem (1). The set of a family of grid points
$[0,1]_\tau \times [0,\pi]_h = \{(t_k, x_n) : t_k = k\tau,\ \ 0 \le k \le N,\ N\tau = 1, x_n = nh, 0 \le n \le M, Mh = \pi\}$
depending on the parameters τ and h is considered for the approximate solution of (1). Here, A is a self-adjoint positive definite operator defined on a Hilbert space H, and f_k, g_k are given smooth functions. Convergence and unconditional stability of linear form of scheme (2) is presented in [9]. In combination with fixed point iteration, scheme (2) gives excellent results in numerical experiments.

NUMERICAL ANALYSIS

In this section we consider a test problem to employ the unified numerical method which combines finite difference method and fixed point iteration. Here, we choose a set of exact solutions

$$u(t,x) = \cos 2t \sin 2x, v(t,x) = \sin 2t \sin 2x, \tag{3}$$

and we formulate an IBVP for coupled Klein-Gordon equations

$$\begin{cases} u_{tt} - u_{xx} = -(u - v) + u^3 + (cos(2t) - sin(2t))\sin(2x) \\ -(cos(2t)\sin(2x))^3, 0 < t < 1, 0 < x < \pi, \\ v_{tt} - v_{xx} + v_t + v = (u - v) + v^3 - (cos(2t) - sin(2t))\sin(2x) \\ -(sin(2t)\sin(2x))^3, 0 < t < 1, 0 < x < \pi, \\ u(0, x) = \sin 2x, u_t(0, x) = 0, 0 \le x \le \pi, \\ v(0, x) = 0, v_t(0, x) = 2\sin 2x, 0 \le x \le \pi, \\ u(t, 0) = u(t, \pi) = 0, 0 \le t \le 1, \\ v(t, 0) = v(t, \pi) = 0, 0 \le t \le 1, \end{cases} \tag{4}$$

which leads to exact solution (3). Applying the first order of accuracy difference scheme (2), the following problem

$$\begin{cases} \frac{{}_m u_n^{k+1} - 2{}_m u_n^k + {}_m u_n^{-k-1}}{\tau^2} - \frac{{}_m u_{n+1}^{k+1} - 2{}_m u_n^{k+1} + {}_m u_{n-1}^{k+1}}{h^2} = -({}_{m-1}u_n^k - {}_{m-1}\, v_n^k) + ({}_{m-1}u_n^k)^3 \\ +(cos(2t_k) - sin(2t_k))\sin(2x_n) + ((cos(2t_k)\sin(2x_n))^3, x_n = nh, 1 \le n \le M-1, \\ \frac{{}_m v_n^{k+1} - 2{}_m v_n^k + {}_m v_n^{-k-1}}{\tau^2} - \frac{{}_m v_{n+1}^{k+1} - 2{}_m v_n^{k+1} + {}_m v_{n-1}^{k+1}}{h^2} = ({}_{m-1}u_n^k - {}_{m-1}\, v_n^k) + ({}_{m-1}v_n^k)^3 \\ -(cos(2t_k) - sin(2t_k))\sin(2x_n) + ((\sin(2t_k)\sin(2x_n))^3, x_n = nh, 1 \le n \le M-1, \\ {}_m u_n^0 = \sin 2x_n, \tau^{-1}({}_m u_n^1 - {}_m\, u_n^0) = 0, x_n = nh, 1 \le n \le M-1, \\ {}_m v_n^0 = 0, \tau^{-1}({}_m v_n^1 - {}_m\, v_n^0) = 2\sin 2x_n, x_n = nh, 1 \le n \le M-1, \\ {}_m u_0^k = {}_m\, u_M^k = 0, {}_m\, v_0^k = {}_m\, v_M^k = 0, 0 \le k \le N \end{cases} \tag{5}$$

is obtained. Here, m is the fixed point iteration index. Rewriting problem (5) with matrix coefficients, an $(N+1)\times(N+1)$ system of linear equations is obtained. The numerical solution is obtained by applying modified Gauss elimination method jointly with fixed point iteration for $m = 1, 2, ..., p$. Here, p depends on a given error tolerance ε such that

$$\left|{}_p u_n - {}_{p-1}\, u_n\right| < \varepsilon, \tag{6}$$

$$\left|{}_p v_n - {}_{p-1}\, v_n\right| < \varepsilon. \tag{7}$$

Numerical results for different values of N and M are presented in tables. The errors are computed by the following formulas:

$$\max_{1 \le k \le N-1, 1 \le n \le M-1} \left|u(t_k, x_n) - u_n^k\right|, \tag{8}$$

$$\max_{1 \le k \le N-1, 1 \le n \le M-1} \left|v(t_k, x_n) - v_n^k\right|. \tag{9}$$

Here, the exact solutions are denoted by $u(t_k, x_n)$, $v(t_k, x_n)$ and the numerical solutions are denoted by u_n^k, v_n^k for the

approximate solution of problem (4) at (t_k, x_n).

TABLE 1. Errors for the approximate solution of problem (4)

First Order of Accuracy Difference Scheme $\left(\varepsilon = 10^{-15}\right)$			
$N = M$	$erroro f u$	$erroro f v$	m
20	0.3280	0.3418	9
40	0.1762	0.1711	10
80	0.0917	0.0858	10
160	0.0470	0.0430	11

TABLE 2. Errors for the approximate solution of problem (4)

First Order of Accuracy Difference Scheme $\left(\varepsilon = 10^{-20}\right)$			
$N = M$	$erroro f u$	$erroro f v$	m
20	0.3280	0.3418	10
40	0.1762	0.1711	11
80	0.0917	0.0858	12
160	0.0470	0.0430	12

In the iterations, the initials are taken as the identity matrices of the form

$$ {}_0u_n^k = I(N+1, M+1), \tag{10} $$

$$ {}_0v_n^k = I(N+1, M+1). \tag{11} $$

The numerical solutions are computed for different values of N and M, by using the first order of accuracy difference scheme and fixed point iteration. The number of iterations increase for small error tolerance ε, and the iteration is stopped when the maximum difference at grid points of two successive results gets less than ε. Note that if the error tolerance ε is changed, the number of iterations alter when the error becomes constant. If N and M are doubled, the value of errors decrease by a factor of 1/2, and this shows the convergence of difference scheme (2).

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