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MAPLE AS A TOOL FOR THE TEACHING OF CYCLOTOMIC POLYNOMIALS ELEMENTARILY

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Abstract. Cyclotomic polynomials are an interesting topic and play an important role for other topics in Number Theory. For special values of n, computing a cyclotomic polynomial is not difficult; this can be done by using properties of the polynomial. For a prime value of n, the polynomial can be written quickly. However, for values which are multiples of odd prime numbers, say $85 = 17 \cdot 5$, the task can be quite difficult if it must be done manually. The polynomial has 41 terms and the degree of 64. Software for cyclotomic polynomials is available; Maple for example, can solve the problem of cyclotomic polynomials very easily. However, understanding of how to compute polynomials is very important for a student in applying various properties of the cyclotomic polynomial. Here, we will use Maple to help students understand cyclotomic polynomials from the basic. Answer is not obtained directly but step-by-step using properties of the polynomial. So, Maple is used to simulate the process of obtaining the polynomial. Once the students grasp the skill, they will be able to use the software for advanced applications.

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1. Introduction

The n-th cyclotomic polynomial, $\Phi_n(x)$, where n is any positive integer, can be defined as the unique irreducible polynomial with integer coefficients. It can also be defined as the minimal polynomial of the n-th primitive roots of unity. Cyclotomic polynomials are an interesting topic and play an important role for other topics in Number Theory such as the factorization of very large numbers; see [3], for example.

Computing $\Phi_n(x)$ is done by using properties of cyclotomic polynomials, which will be described in the following part of this paper. For special values of n, computing the cyclotomic polynomial is not difficult. For a prime value of n, even a big one such as 65537 (which is the fourth Fermat prime), the polynomial can be computed easily. For a modest value of n, such as 53, we can write all the terms of the polynomial quickly. For other values, e.g., for multiples of several integers such as 85, which is $17 \cdot 5$, or 105, which $3 \cdot 5 \cdot 7$, the task can be quite difficult if it must be done manually. But at least undergraduate students who are studying cyclotomic polynomials are expected to be able to compute $\Phi_n(x)$ manually for $n \leq 50$ or when n is a prime (which we call simple values in this paper). For higher

values of n the computation of $\Phi_n(x)$ is more difficult except for particular values of n which will be described in the following part. Symbolic computation software such as Maple or Mathematica helps us in doing this. For a good treatment on symbolic computation in general, see [10], for example. Examples of the use of Maple and Mathematica in scientific computing and engineering can be found in [12], [15] and [24].

2. Important properties of cyclotomic polynomials

The n-th cyclotomic polynomial, $\Phi_n(x)$ is defined by

$$\Phi_n(x) = \prod_{\substack{j=1 \\ \gcd(j,n)=1}}^{n} (x - e^{2\pi ij/n}) = \sum_{k=0}^{\varphi(n)} a_n(k)x^k, \tag{1}$$

where n and j are positive integers and φ is Euler's totient function. $a_n(k)$ is the coefficient of the cyclotomic polynomial; it is 0 if $k > \varphi(n)$. See [3], [4] and [7] for the basic formulas for cyclotomic polynomials.

The values of $\varphi(n)$ can be computed using the following properties:

$$\varphi(1) = 1, \tag{2}$$

$$\varphi(p) = p - 1, \quad \text{if } p \text{ is a prime}, \tag{3}$$

$$\varphi(n) = n \prod_{p|n} (1 - 1/p), \tag{4}$$

$$\varphi(p_1 p_2) = \varphi(p_1)\varphi(p_2), \quad \text{if } p_1 \text{ and } p_2 \text{ are distinct primes}. \tag{5}$$

Formula (5) can be extended as follows. If $p_1, p_2, \ldots, p_k$ are distinct primes, we then have

$$\varphi(p_1 p_2 \cdots p_k) = \varphi(p_1)\varphi(p_2)\cdots\varphi(p_k), \quad \text{i.e.,}$$
$$\varphi(p_1 p_2 \cdots p_k) = (p_1 - 1)(p_2 - 1)\cdots(p_k - 1). \tag{6}$$

Formulas (4) and (6) are equivalent for $n = p_1 p_2 \cdots p_k$. For detailed description of Euler's totient function, see [1], [13], [15] or [16].

Next, we can compute $\Phi_n(x)$ using the following property (see also [3], [4] and [7]):

$$\Phi_n(x) = \frac{x^n - 1}{\prod_{d|n, d<n} \Phi_d(x)}. \tag{7}$$

The first cyclotomic polynomial is $\Phi_1(x) = x - 1$. If p is a prime, we simply have

$$\Phi_p(x) = x^{p-1} + x^{p-2} + \cdots + x + 1. \tag{8}$$

For example, $\Phi_7(x) = x^6 + x^5 + x^4 + x^3 + x^2 + x + 1$. If $n = 2p$ where p is an odd integer, we then have

$$\Phi_{2p}(x) = \Phi_p(-x). \tag{9}$$

If $n = p^m$ where p is a prime, then

$$\Phi_n(x) = \Phi_p(x^k), \quad \text{where } k = p^{m-1}. \tag{10}$$

In general, if $n = p^m r$ and r is relatively prime to p, we then have

$$\Phi_n(x) = \Phi_{pr}(x^k), \quad k = p^{m-1}. \tag{11}$$

Formulas (8) to (11) are also given in [3], [4] and [7].

The following relationship is very useful:

$$\Phi_1(x)\Phi_p(x) = x^p - 1; \tag{12}$$

it can be proved very easily (just multiply Eq. (8) by $x - 1$). By using Eq. (7) repeatedly we can deduce that

$$\Phi_{pqr}(x) = \frac{\Phi_p(x^{qr})\Phi_p(x)}{\Phi_p(x^q)\Phi_p(x^r)}, \tag{13}$$

where p, q and r are distinct primes. This formula is very useful to compute $\Phi_n(x)$ where n is the multiple of three distinct primes; see [18] and [25], for example.

The degree of $\Phi_n(x)$ can be easily found by using Euler's totient function $\varphi(n)$. The following are cyclotomic polynomials for various "simple values" of n:

$\Phi_2(x) = x + 1 \leftarrow 2$ is prime, use Eq. (8);

$\Phi_3(x) = x^2 + x + 1 \leftarrow 3$ is prime, use Eq. (8);

$\Phi_6(x) = \Phi_3(-x) = x^2 - x + 1 \leftarrow 6 = 2 \cdot 3$, 3 is prime; use Eq. (9) and $\Phi_3(x)$;

$\Phi_4(x) = \Phi_2(x^2) = x^2 + 1 \leftarrow 4 = 2^2$, 2 is prime; use Eq. (10) and $\Phi_2(x)$;

$\Phi_{27}(x) = \Phi_3(x^9) = x^{18} + x^9 + 1 \leftarrow 27 = 3^3$, 3 is prime; use Eq. (10) and $\Phi_3(x)$.

We can easily compute the n-th cyclotomic polynomial for $n = 997$ (which is prime) or $n = 707\,281$ (which is 29^4) although writing them completely will take a lot of space. However, it is more difficult to compute $\Phi_{15}(x)$ or $\Phi_{77}(x)$.

We will now compute $\Phi_{15}(x)$. Since $15 = 3 \cdot 5$, all positive divisors of 15 are 1, 3 and 5. From Eq. (7) we have

$$\Phi_{15}(x) = \frac{x^{15} - 1}{\Phi_1(x)\Phi_3(x)\Phi_5(x)}.$$

From Eq. (12), $\Phi_1(x)\Phi_5(x) = x^5 - 1$, so

$$\Phi_{15}(x) = \frac{x^{15} - 1}{\Phi_3(x)(x^5 - 1)} = \frac{x^{10} + x^5 + 1}{x^2 + x + 1}.$$

Using synthetics division, we finally have $\Phi_{15}(x) = x^8 - x^7 + x^5 - x^4 + x^3 - x + 1$. The degree of $\Phi_{15}(x)$ can be found using Eq. (4) or Eq. (6). From Eq. (4), $\varphi(15) = \varphi(3 \cdot 5) = 15(1 - \frac{1}{3})(1 - \frac{1}{5}) = 8$. Or, from Eq. (6), $\varphi(3 \cdot 5) = (3 - 1)(5 - 1) = 8$.

Now, for more difficult problems such as computing $\Phi_{105}(x)$ or $\Phi_{1001}(x)$ we will resort to software to help us, which will be discussed in the following sections.

Last but not least, the difficulty of computing $\Phi_n(x)$ does not necessarily depend on the size of n but also on its properties. Computing $\Phi_{65537}(x)$ is far much easier compared to that of $\Phi_{105}(x)$ since 65537 is a prime number!

3. Coefficients of cyclotomic polynomials

Coefficients of all cyclotomic polynomials for $n < 105$ are in $\{-1, 0, 1\}$; this fact is mentioned in almost any paper on cyclotomic polynomials, see [6], [7] or [18], for example. For proof, see [7]. 105 is the multiple of the three smallest odd primes (3, 5 and 7) and it is the smallest number such that some coefficients of the corresponding cyclotomic polynomial are not in $\{-1, 0, 1\}$.

For $n < 385$, the coefficients do not exceed 2 in absolute value; see [20], for example. Coefficients of $\Phi_{385}(x)$ are in $\{\pm 1, \pm 2, -3\}$. It has 177 terms; the degree of this polynomial is 240. The computation for $\Phi_{385}(x)$ will be made in Section 5 of this article.

Schur's theorem is important: *There exist cyclotomic polynomials with coefficients arbitrarily large in absolute value.* However, Schur himself did not prove this theorem; it was proved by Lehmer [20]. Arnold and Monagan [2] provide a table for $A(n)$ for n as a product of the smallest odd primes, where $A(n)$ is the maximum absolute value of the coefficients of $\Phi_n(x)$.

A lot of research has been devoted to compute maximum values of coefficients for cyclotomic polynomials; see [4], [10], [17], [22] and [24], for more detailes. Zhao and Zhang [25] provide analysis for the coefficients of the cyclotomic polynomials of order three. Bachman [4] gives an asymptotic formula for $\log a(m)$ where $a(m) = \max_n |a(m, n)|$ and $a(m, n)$ denotes the m-th coefficient of the n-th cyclotomic polynomial $\Phi_n(x)$, that improves a previous estimate of Montgomery and Vaughan. In [10], it is mentioned t hat Gallot and Moree in [8] disproved the Bieter conjecture which asserts that if $p < q < r$ are primes, then $|a_{pqr}(k)| \leq \frac{p+1}{2}$.

4. Introduction to Maple, a symbolic computation language

Maple is a very powerful symbolic computation language. It was developed in the University of Waterloo, Canada. Differently from conventional programming languages, symbolic computation languages can accept inputs symbolically. Outputs can be also produced symbolically. These features are very useful when we want to derive a formula or to prove a theorem, for example. Moreover, a symbolic programming language can be used for computation that needs multi-precision arithmetic. We can easily compute a value to thousands of digits very quickly. This feature is often required in number theory.

Maple has many built-in functions for many areas of mathematics such as number theory. We can compute a formula (function) directly by calling its name. Often, we must call a library first where a formula is stored before it can be called. There are many libraries in Maple with thousands of functions. The Num theory is a Maple's library which must be called if we want to compute a cyclotomic polynomial. In this paper, we will use Maple V, which still needs a user to write

a command. Newer versions are much easier to use because we can just click a formula instead of writing its name. We write as follows:

```
> restart;
> with(Num theory);
```

Various choices will be shown. What we need is `cyclotomic`. Suppose we want to compute $\Phi_7(x)$, so write

```
> cyclotomic(7, x);
```

Maple will produce the following output instantly:

$$x^6 + x^5 + x^4 + x^3 + x^2 + x + 1.$$

```
> cyclotomic(14, x);
```

$$x^6 - x^5 + x^4 - x^3 + x^2 - x + 1.$$

This result can be obtained easily using Eq. (9) where $\Phi_{14}(x) = \Phi_7(-x)$.

We can try a very big value, say $n = 1879$. Just very quickly the result is displayed in the screen:

$$x^{1878} + x^{1877} + x^{1876} + \cdots + x^2 + x + 1.$$

The whole output contains 1879 terms! Here 1879 is a prime.

For a good introduction to Maple refer to Maple Manual such as [12] or [21]. [12] is also very useful for an introduction of Maple to mathematics; the book is very readable and has many examples. There are many textbooks which provide applications of Maple in mathematics, physics or engineering. For applications of Maple in mathematics, see [5], [14] and [23]. Advanced applications of mathematics are shown in [19].

5. Using Maple to simulate the process of computing cyclotomic polynomials

Since Maple can compute $\Phi_n(x)$ very easily even for a very large value of n, why do we need to simulate the process to compute it? As a problem solver, Maple is very good at its task. It will try to solve problems provided by a user. However, there is no explanation how it solves a problem. In a learning process, it is very important that students master a subject they are studying. This goal cannot be achieved if they just want to get the final result without bothering how to find it. By applying various properties for the cyclotomic polynomial, they will be able to compute $\Phi_n(x)$ manually for simple values of n. Here, we will tailor Maple to help students understand cyclotomic polynomial from the basic. Answer is not obtained directly but step-by-step using properties for the polynomial. So, Maple will be used to simulate the process of obtaining the polynomial for reasonably large numbers. Once the students grasp the skill, they will be able to use the software for advanced applications. Next, we will give examples how Maple can be used to

help students understand cyclotomic polynomials better. Several commands and functions used are explained. Our comments are put inside curly brackets.

After we open Maple (by clicking the icon Maple on the screen), we write the following commands:

```
> restart;
> with(Num theory):
> cyc[1] := x-1; # definition for n=1
> cycpr := proc(n,x) # Eq. (8)
```

{ `proc` is a procedure in Maple, which is like a function in other programming languages.}

```
> local sn,i;
> sn := 0; if is prime(n) then for i from 1 to n do sn := sn +
x^(i-1); od; fi; end;
> cyc2pr := proc(n,x) # Eq. (9)
> local sn,i,m;
> sn := 0; m := n/2; if is prime(m) then for i from 1 to m do sn :=
sn + (-1)^(i-1)x^(i-1); od; fi; end;
> fn := (x,n) -> x^n-1; # Eq. (12)
> fpm := (p,m,x) -> x^(p^(m-1)); # Eq. (10)
```

Now, we will give three examples how to use the Maple statements to compute cyclotomic polynomials. Suppose we want to compute $\Phi_{88}(x)$. We know that $88 = 2^3 \cdot 11$; so we use Eq. (11). Here, $p = 2$, $m = 3$, $r = 11$ and $k = p^{m-1} = 4$, $\Phi_{88}(x) = \Phi_{2\cdot 11}(x^4)$. In this case we have to compute $\Phi_{22}(x)$ first by calling `cyc2pr(22, x)`:

```
> cyc[22] := cyc2pr(22,x); # This produce Φ22(x):
```

$$\texttt{cyc[22]} := x^{10} - x^9 + x^8 - x^7 + x^6 - x^5 + x^4 - x^3 + x^2 - x + 1.$$

{we can easily find cyc[2] since $22 = 2 \cdot 11$ and 11 is a prime}

Before computing $\Phi_{22}(x^4)$, we change cyc[22] to a function using `unapply` as follows:

```
> cyc[22] := unapply(cyc[22],x);
```

$$\texttt{cyc[22]} := x \to x^{10} - x^9 + x^8 - x^7 + x^6 - x^5 + x^4 - x^3 + x^2 - x + 1.$$

Now we can find $\Phi_{88}(x)$:

```
> cyc[88] := cyc[22](x^4)
```

$$\texttt{cyc[88]} := x^{40} - x^{36} + x^{32} - x^{28} + x^{24} - x^{20} + x^{16} - x^{12} + x^8 - x^4 + 1.$$

{instead of changing `cyc[22]` into a function, we can also find `cyc[88]` using the following command:

```
cyc[88]:=subs(x=x^4,cyc[22]).
```

Here, `subs` is substitution. The result will be the same.}

We can of course compute $\Phi_{88}(x)$ directly as follows.

```
> cyclotomic(88, x);
```

similar to the example in Section 3. However, this will not enhance the students' ability to understand cyclotomic polynomials. We will use Maple's command directly if we want to check our result.

The next example is the computation of $\Phi_{21}(x)$. Since $21 = 1 \cdot 3 \cdot 7$, we have, using Eq. (7),

$$\Phi_{21}(x) = \frac{x^{21} - 1}{\Phi_1(x)\Phi_3(x)\Phi_7(x)}.$$

But $\Phi_1(x)\Phi_7(x) = x^7 - 1$, hence

$$\Phi_{21}(x) = \frac{x^{21} - 1}{\Phi_3(x)(x^7 - 1)} = \frac{x^{14} + x^7 + 1}{\Phi_3(x)}.$$

Multiplying both numerator and denominator by $x - 1$ produces

$$\Phi_{21}(x) = \frac{x^{15} - x^{14} + x^8 - x^7 + x - 1}{x^3 - 1}.$$

Of course, we do not necessarily do these steps by ourselves. Using Maple, we have

```
> f1 := fn(x,21):  # we just suppress the output x^21-1
> f2 := expand(cypr(x,7)(x-1)):  # the output is x^7-1
> f3 := simplify(f1/f2):  # the output is x^14+x^7+1
> f4 := expand(f3(x-1)):  # what is the output?
> f5 := expand(cypr(x,3)(x-1)):  # the output is x^3-1
> cyc[21] := simplify(f4/f5);
```

$$\texttt{cyc[21]} := x^{12} - x^{11} + x^9 - x^8 + x^6 - x^4 + x^3 - x + 1.$$

We can check the highest degree using $\varphi(n)$. Here $\varphi(21) = 21(1 - \frac{1}{3})(1 - \frac{1}{7}) = 12$. There is a command for $\varphi(n)$ in Maple that is `phi(n)`.

```
> phi (21); # the result is 12
```

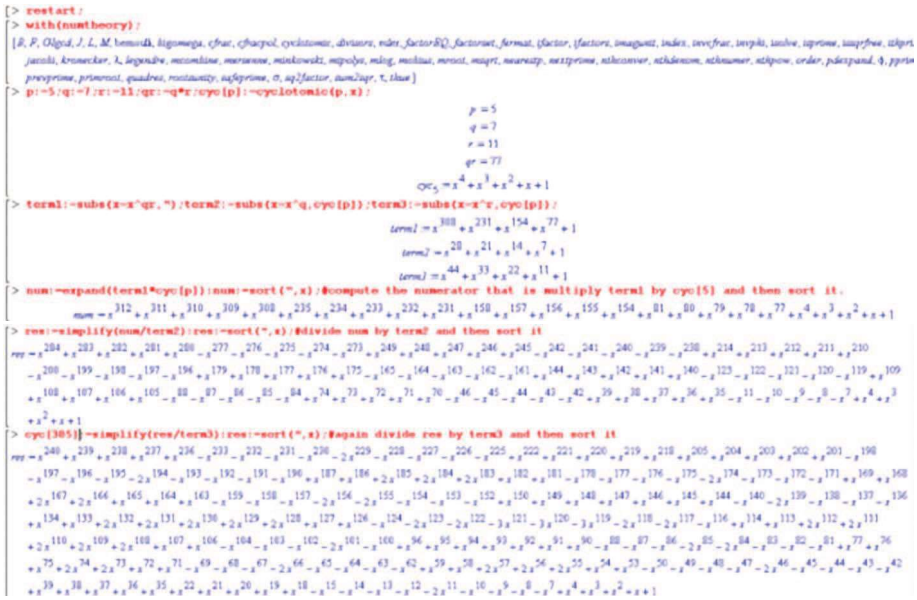
The last example will be the computation of $\Phi_{385}(x)$. Since $385 = 5 \cdot 7 \cdot 11$ is a multiple of three distinct primes, we will use Eq. (13) to compute the cyclotomic polynomial. Here, $p = 5$, $q = 7$, and $r = 11$. So, $qr = 77$. We then have

$$\Phi_{385}(x) = \frac{\Phi_5(x^{77})\Phi_5(x)}{\Phi_5(x^7)\Phi_5(x^{11})}. \tag{14}$$

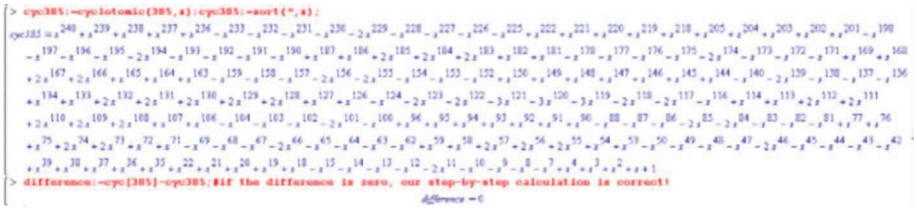
We can compute each term in aforementioned formula easily. Here,

$$\begin{aligned}
\texttt{cyc[5]} &= \Phi_5(x) = x^4 + x^3 + x^2 + x + 1,\\
\texttt{term[1]} &= \Phi_5(x^{77}) = x^{308} + x^{231} + x^{154} + x^{77} + 1,\\
\texttt{term[2]} &= \Phi_5(x^7) = x^{28} + x^{21} + x^{14} + x^7 + 1,\\
\texttt{term[3]} &= \Phi_5(x^{11}) = x^{44} + x^{33} + x^{22} + x^{11} + 1.
\end{aligned}$$

Polynomial division is best done by Maple. Doing this manually is tedious and prone to error, particularly when it involves high degrees. Since the computation is quite complex, we will resort to Maple to help us to compute the righthand side of Eq. (14). We will do it step by step using all terms in Eq. (14). Screenshots of Maple's programs are shown as follows.



Now, we will ask Maple to compute `cyc[385]` directly and then verify our result:



Since the computed cyclotomic polynomials have a large number of coefficients, it is difficult to see if they are the same. Now, by just subtracting the first from the other, we can easily verify that they are similar if their difference is zero.

But what is the degree of the polynomial? We know it is $\varphi(385)$ which is $385(1-\frac{1}{5})(1-\frac{1}{7})(1-\frac{1}{11}) = 240$. We can also find it using the following command

```
> degree := degree(");
```

$$\texttt{degree} := 240$$

If we do not sort the polynomials, the degree of the first coefficient may not be 240. That is why we sort them before displaying the result.

The coefficients are in $\{\pm 1, \pm 2, -3\}$. We can find the number of terms using

```
> terms := nops("");
```

$$\texttt{terms} := 177$$

We can ask Maple to determine how many terms are there for each value as follows:

```
> sum1 := 0:  for i from 240 by -1 to 0 do
> if ck[i] = -1 then sum1 := sum1 + abs(ck[i]):  fi; od; sum1; 68
```

The aforementioned codes ask Maple to find the number of terms whose values are -1 each. The answer is 68. Likewise, by slightly modifying the codes we can determine the number of other terms. The results are shown in Table 1.

Value	-3	-2	-1	1	2
Number of terms	3	16	68	70	20

Table 1. Number of terms for each value of the coefficients of Φ_{385}

6. Conclusion

While Maple can compute the n-th cyclotomic polynomial very easily, the direct use of Maple will not help students understand the computation of the polynomial because they just get the answer but do not know from where the answer comes. By simulating the computation process using Maple, students will get better understanding of the process. They will gain knowledge and skill, which they will not if they just apply the software directly. Once they grasp the skill, they will be able to use the software for advanced applications.

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